A Study on the Confidence Limit for Two Independent Probability Variables in Engineering Problems*
(Applications to Limit of Transmitted Torque in Disk Clutch and Bolt Axial Tension Control)

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The problem of the product of two independent probability variables which are normally distributed was theoretically analyzed using the confidence limit ellipse. By applying $z=xy$ type problems to a disk clutch, the limit of transmitted torque was rationally calculated. On the axial tension control in bolted joints for the $z=x/y$ type problem, experimental analysis concerning bolt tightening by the calibrated wrench method was carried out under dry, oil and anaerobic adhesive conditions. For this type of problem, the proposed method is applicable in the case in which the hyperbolic relation can be treated linearly. For the distribution of product z, the method for the calculation of probability P_f, when z did not exceed a limit, was shown. From the result of the analysis by the proposed method, it was found that the maximum and minimum values in the scatter of the product z by the conventional method resulted in the use of a higher confidence limit level that corresponded to $\sqrt{2}$ times greater in the percentile value in the standard normal distribution table.

Key Words: Machine Element, Fixing Element, Reliability Engineering, Confidence Limit, Probability Density Function, Normal Distribution, Ellipse of Confidence Limit, Disk Clutch, Transmitted Torque, Bolted Joint Tightening, Axial Tension Control

1. Introduction

In mechanical engineering, a reliable design approach should be taken by regarding various physical quantities as probability variables. Particularly, this kind of approach is quite important for the problems related to quality control.

For example, in bolt tightening, as shown in Fig. 1, the torque T given to a bolt is determined as the product of given force F and the distance L to the point where the force is applied. The force to be given is scattered and the position where the force is applied also scatters at each tightening, as typically shown in Fig.1. Accordingly, the torque applied to a bolt has scattering. The maximum tightening torque is calculated by taking the maximum values of the scattering of the force F and distance L into consideration.

So far, the maximum value T_{max} of the torque applied to a bolt has been determined by the product of maximum value F_{max} in the scattering of given force and maximum value L_{max} in that of the distance, as in Eq.(1).

$$T_{max}=F_{max}\cdot L_{max} \qquad (1)$$

However, when a certain confidence limit is considered, T_{max} is not determined in the simple form as in Eq.(1), but no research on this kind of probability distribution and confidence limit in engineering problems has been carried out except for the papers by Yoshimoto[1] and those by the authors[2][3].

This report is an extension of the previous papers.

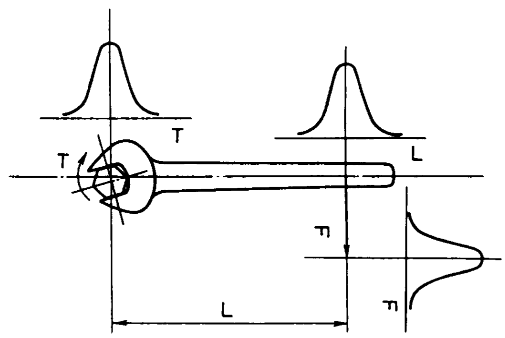

Fig. 1 Tightening work by wrench

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It is well known that the product of two normally distributed variables is also normally distributed.

This concept is applied to obtain the limits of transmitted torque for a disk clutch. In the case of a bolt, the probability variables are not the product of two normally distributed variables but the quotient. However, under certain conditions, it is possible to apply the same kind of analysis.

2. Theoretical Analysis

Two independent probability variables are denoted by x and y, and their product z is defined as in Eq.(2).

$$z=xy \qquad (2)$$

Now, it is assumed that the product z is also a probability variable, and all of the three probability variables, x, y and z, are normal distributions. Accordingly, the probability density functions and normal distributions of x, y and z are shown in Table 1.

When Eq.(2) is shown by probability density functions, Eq.(3) is obtained.

$$h(z)=f(x)\cdot g(y) \qquad (3)$$

Since $f(x)$ and $g(y)$ are considered to be mutually independent, $h(z)$ becomes a coupling probability density function. $h(z)$ can be shown as below.

$$h(z)=\frac{1}{\sqrt{2\pi}\sigma_z}\exp\left\{-\frac{1}{2}\frac{(z-\mu_z)^2}{\sigma_z^2}\right\} \qquad (4)$$

On the other hand, the right side of Eq.(3) is as below.

$$f(x)\cdot g(y)=\frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\left[\frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\} \qquad (5)$$

Now, as mentioned above, since it was considered that the product of $f(x)$ and $g(y)$ becomes a coupling probability density function, and is a normal distribution, Eqs.(4) and (5) must be equivalent for all the values of x, y and z. Therefore, the following equations are obtained.

$$\sigma_z=\sqrt{2\pi}\sigma_x\cdot\sigma_y \qquad (6)$$

$$\frac{(z-\mu_z)^2}{\sigma_z^2}=\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2} \qquad (7)$$

Table 1 List of probability variables and functions

Variable	pdf.	Normal distribution
x	f(x)	$N(\mu_x, \sigma_x^2)$
y	g(y)	$N(\mu_y, \sigma_y^2)$
z	h(z)	$N(\mu_z, \sigma_z^2)$

pdf. :Probability density function

Since x and y are mutually independent, in order to satisfy Eq.(7) for x, y and z, $x=\mu_x$ and $y=\mu_y$ when $z=\mu_z$. From the relation of these values and Eq.(2), the following relationship is obtained.

$$\mu_z=\mu_x\cdot\mu_y \qquad (8)$$

Here, variable z is shown by Eq.(9).

$$z=\mu_z+r\sigma_z \qquad (9)$$

where r is a variable when z is shown by μ_z and σ_z (percentile valve in normal distribution).

Then, Eq.(7) becomes as follows.

$$\frac{(x-\mu_x)^2}{A^2}+\frac{(y-\mu_y)^2}{B^2}=1 \qquad (10)$$

where $A=r\sigma_x$

$B=r\sigma_y$.

This shows the ellipse of the confidence limit, as shown in Fig. 2. An arbitrary point on the ellipse is denoted by $p(x,y)$. Thus, z, the product of x and y from Eq.(2), is given by the area enclosed by both coordinate axes and the two straight lines passing through this point $p(x,y)$ parallel to them. Using the angle θ in the figure, z is expressed by $Z(\theta)$ as follows.

$$Z(\theta)=(\mu_x+A\cdot\cos\theta)(\mu_y+B\cdot\sin\theta) \qquad (11)$$

For determining the maximum value of $Z(\theta)$, it will suffice to differentiate Eq.(11) with θ and determine the values of θ that give the maximum and minimum of $Z(\theta)$.

$$Z'(\theta)=-A(\mu_y+B\cdot\sin\theta)\sin\theta+B(\mu_x+A\cdot\cos\theta)\cos\theta \qquad (12)$$

In this case, it is obvious that point , which gives the maximum value of $Z(\theta)$ exists in the first quadrant of the coordinates of the ellipse in the figure, and from the form of the function, $Z(\theta)$ becomes a gradually decreasing function in the range of θ from 0° to 90°, and has only one solution. This solution, θ_z, gives the point at which $Z(\theta)$ becomes maximum. On the other hand, it is also obvious that the point which gives the minimum value of $Z(\theta)$ is at the position symmetrical to the center of the ellipse ; therefore, the maximum value Z_{max} and minimum value Z_{min} of $Z(\theta)$

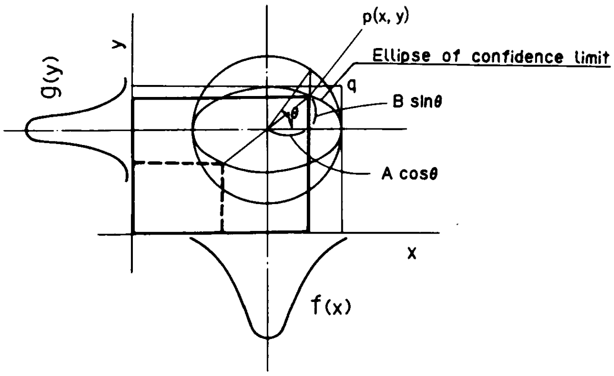

Fig. 2 Distribution for product of x and y

can be shown as follows.

$$Z_{max}=(\mu_x+A\cdot\cos\theta_z)(\mu_y+B\cdot\sin\theta_z) \quad (13)$$

$$Z_{min}=(\mu_x-A\cdot\cos\theta_z)(\mu_y-B\cdot\sin\theta_z) \quad (14)$$

Since it is assumed that z is also a normal distribution, from Eqs.(13) and (14), the mean value and standard deviation of $Z(\theta)$ are determined as follows.

$$\mu_z=(\mu_x\mu_y+r^2\sigma_x\sigma_y\cdot\sin\theta_z\cdot\cos\theta_z) \quad (15)$$

$$\sigma_z=(\mu_x\sigma_y\cdot\sin\theta_z+\mu_y\sigma_z\cdot\cos\theta_z) \quad (16)$$

In applying this concept to mechanical design as in failure problems, the limit value should be considered as probability function. For example, in the yield point design, the upper limit S_u is used, and in bolted joint design against loosening, the lower limit S_l is used, as shown in Fig. 3.

In the case of the upper limit, when z is not greater than the upper limit S_u, probability P_{fu} is obtained as follows.

$$P_{fu}=P_{rob}(z\leqq S_u) \quad (17)$$

And Eq.(17) is shown as follows.

$$P_{fu}=P_{rob}(u\leqq 0)=\int_{-\infty}^{\infty}f_u(u)du \quad (18)$$

when $u=S_u-z$.

The distribution of the difference u is also the normal distribution which consists of mean value $(\mu_{su}-\mu_z)$ and standard deviation $\sqrt{\sigma_{su}{}^2+\sigma_z{}^2}$, when S_u and z are mutually independent and are normal distributions.

As is well known, standard normal distribution $F(t)$ is obtained as the following equation.

$$F(t)=\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt\equiv\Phi(t) \quad (19)$$

where

$$t=\frac{\{u-(\mu_{su}-\mu_z)\}}{\sqrt{\sigma_{su}^2-\sigma_z^2}}.$$

Thus, the probability P_{fu} is obtained as follows.

$$P_{fu}=\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi(\sigma_{su}^2+\sigma_z^2)}}\exp\left[-\frac{\{u-(\mu_{su}-\mu_z)\}^2}{2(\sigma_{su}^2+\sigma_z^2)}\right]du$$

$$=\Phi\left(-\frac{\mu_{su}-\mu_z}{\sqrt{\sigma_{su}^2+\sigma_z^2}}\right)=1-\Phi\left(\frac{\mu_{su}-\mu_z}{\sqrt{\sigma_{su}^2+\sigma_z^2}}\right) \quad (20)$$

where μ_{su}: mean value at the distribution of upper limit

σ_{su}: standard deviation.

On the other hand, in the case of the lower limit,

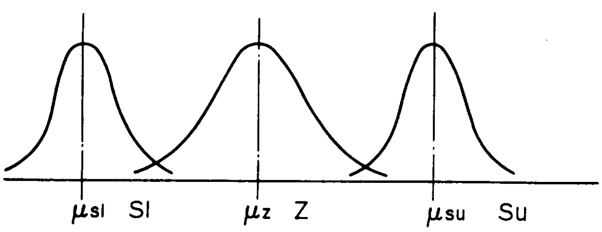

Fig. 3 Distribution of z and limits

when z is not less than the lower limit S_l, the probability P_{fl} is obtained as follows.

$$P_{fl}=1-\Phi\left(\frac{\mu_z-\mu_{sl}}{\sqrt{\sigma_z^2+\sigma_{sl}^2}}\right) \quad (21)$$

where μ_{sl}: mean value at the distribution of lower limit

σ_{sl}: standard deviation.

As described above, when the probabilities P_{fu} and P_{fl} are known, the reliability design in mechanical engineering problems expressed by equation $z=xy$ can be treated.

3. Application to Problems of Mechanical Engineering

3.1 Application to the limit of transmitted torque for a disk clutch

In automobiles and industrial vehicles, friction clutches are frequently used for transmitting or cutting off power. Figure 4 shows a single disk-type friction clutch used for forklifts. Torque M_c transmitted through a clutch is determined by the following equation.

$$M_c=z_f\cdot\mu_c\cdot r_m\cdot F \quad (22)$$

where

z_f : number of friction surfaces,

μ_c : coefficient of friction of clutch surfaces

F : pressing force at the time of engaging clutch

r_m : mean radius $r_m=(r_1+r_2)/2$; where r_1 and r_2 are outer and inner radii of disk clutch, respectively.

Now, in this case, it is considered that from the geometry of the clutch, the number of frictional surfaces and mean radius can be treated as constants. Thus, by setting $c'=z_f.\ r_m$, Eq.(22) becomes.

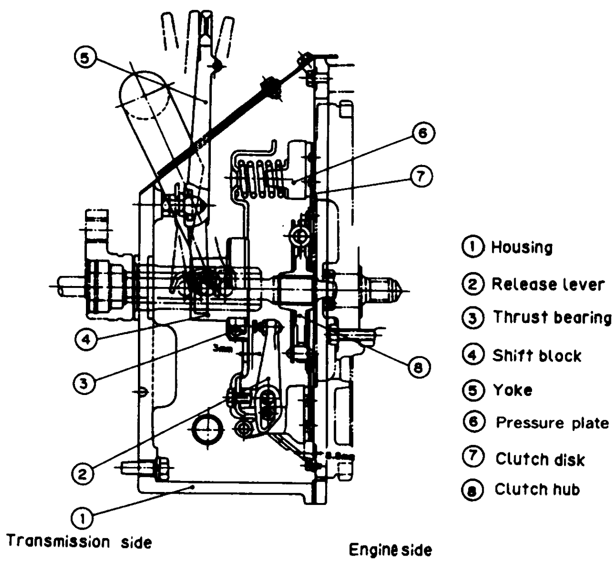

Fig. 4 Disk cluch for forklift truck

$$M_c = c' \cdot \mu_c \cdot F \qquad (23)$$

The coefficient of friction of a clutch and the pressing force of springs at the time of engaging are normal distributions, as seen in Table 2, according to the experimental results of the testing of a large number of clutch units. In addition, it has been confirmed that the limit of transmitted torque M_c is also a normal distribution.

The coefficient of friction depends on the material characteristics of the engaged surfaces; on the other hand, the spring force is influenced by the scattering of dimensions, such as the diameter and pitch in manufacturing or the scattering of the setting length in the clutch. It is considered that both can be treated as independent probability variables.

Accordingly, by directly utilizing the method described in section 2, the degree of scattering of the transmitted torque can be reasonably shown as a probability value by the analysis using a confidence limit ellipse.

By setting the confidence limit value to 90% and using the successive approximation method for Eq. (12), the maximum angle of the confidence limit for the limit of transmitted torque was obtained as 88.6°. This result was shown according to Eq. (10) in Fig. 5. Table 3 shows the summary of these results of analysis.

3.2 Application to the axial tension control in bolts

The method shown in section 2 can be applied to problems of the $z=x/y$ type under special conditions. This example is related to the axial tension control in tightening a bolt by the calibrated wrench method. In this section, the results of the experimental analysis of the confidence limit of the axial tension distribution in the tightening metric fine thread in the dry condition, in the oil lubricated condition and by using anaerobic adhesives Loctite 262 are reported.

From Fig. 6, the basic equation for tightening a bolt by the calibrated wrench method is

$$T = KPd \qquad (24)$$

where

P : axial tension
T : tightening torque
K : torque coefficient
d : nominal diameter.

The axial tension distribution after tightening bolts is expressed using Eq. (24) as follows.

$$P = \left(\frac{1}{K}\right)\left(\frac{T}{d}\right) = k \cdot T' \qquad (25)$$

where

k : axial tension coefficient ($k=1/K$)
T' : unit tightening torque ($T'=T/d$).

In most cases of tightening bolts, the scattering of axial tension coefficient k is determined essentially by the scattering of the coefficient of friction of screw thread surfaces and the geometry of screws. On the other hand, as shown in Fig. 1, the scattering of tightening torque T is determined by the point where force is applied to a calibrated wrench and the force to be given. It arises from the individuality of the workers, the condition of tools and so on. Accordingly, it is reasonable to consider that K and T are independent

Table 2 List of probability variables in the case of transmitted torque at disk cluch

Variable	Normal distribution
μ_c	N(0.40, 0.0304²)
F(KN)	N(5.25, 0.159²)

Table 3 Result of analysis

Term		Result
θ_A(°)		88.6
M_c (N·m)	max	248.7
	mean	236.2
	min	223.6

c' = 112.5mm

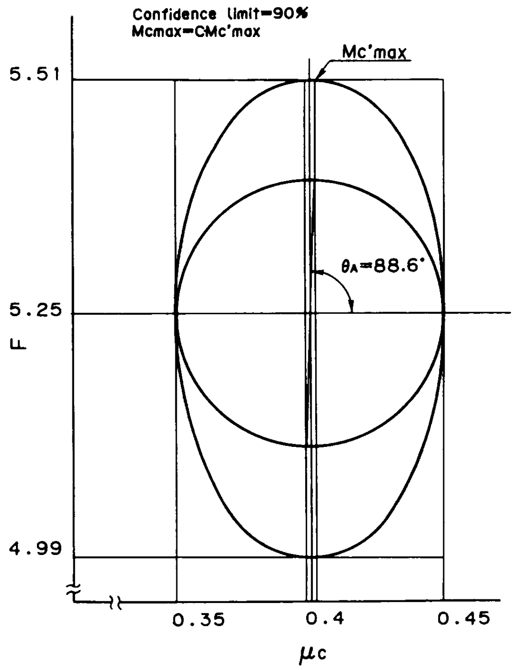

Fig. 5 Limit transmitted torque by confidence limit

probability variables.

Now, it is assumed that k, T' and P are probability density functions, as shown in Table 4, and are normal distributions.

Tool coefficient a (expressing the scattering of tightening torque due to differences among tightening tools or workers) is shown as follows.

$$a=\frac{T_{\max}-T_{\min}}{2T_{\text{mean}}} \quad (26)$$

$$\mu_T=T_{\text{mean}} \quad (27)$$

$$\sigma_T=\frac{a}{r}T_{\text{mean}} \quad (28)$$

From Eq.(25), the relation between axial tension coefficient k and torque coefficient K becomes a hyperbola. Usually, torque coefficient K is about 0.2, and in the range around this value the relationship can be treated linearly from the form of the hyperbola ; axial tension coefficient k also is a normal distribution if K is a normal distribution. Then axial tension coefficient k is expressed by Eq.(29) in practical use.

$$k=\frac{-(k_{\max}-k_{\min})K+(K_{\max}\cdot k_{\max}-K_{\min}\cdot k_{\min})}{K_{\max}-K_{\min}} \quad (29)$$

Table 4 List of probability variables in the case of axial tension control on tightening bolt.

Variable	pdf.	Normal distribution
k	f(k)	N(μ_K , $\sigma_K{}^2$)
T'	g(T')	N(μ_T , $\sigma_T{}^2$)
P	h(p)	N(μ_P , $\sigma_P{}^2$)

pdf. :Probability density function

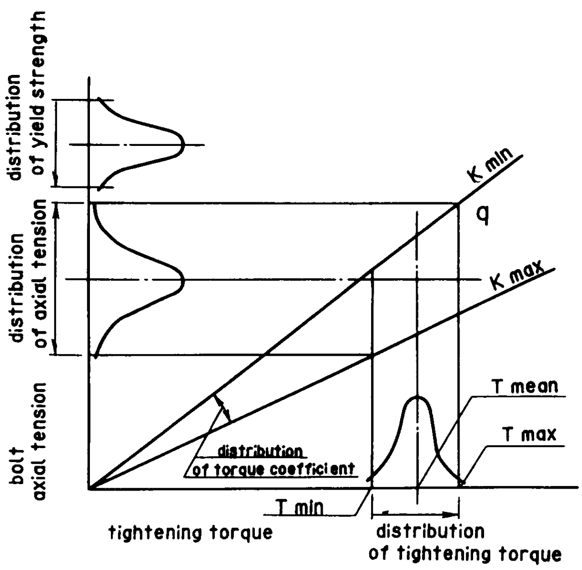

Fig. 6 Relationship between tightening torque and axial tension

Mean value μ_k and standard deviation σ_k of k are obtained as follows.

$$\mu_k=\frac{K_{\max}+K_{\min}}{2K_{\max}\cdot K_{\min}} \quad (30)$$

$$\sigma_k=\frac{K_{\max}-K_{\min}}{rK_{\max}\cdot K_{\min}} \quad (31)$$

Thus, Eq.(12) is expressed as follows.

$$Z'(\theta)=\left[-\left\{\frac{(K_{\max}+K_{\min})}{2}+(K_{\max}-K_{\min})\sin\theta\right]\sin\theta+\left\{(K_{\max}-K_{\min})\left(\frac{1}{a}+\cos\theta\right)\right\}\cos\theta\right]\times\frac{aT_{\text{mean}}}{d\cdot K_{\max}\cdot K_{\min}} \quad (32)$$

In this case, solution θ_z is determined uniquely only by knowing the physical properties and probabilistic statistical properties of the screws and tools, namely the torque coefficient and tool coefficient. θ_z is independent of the value of tightening torque and dimensions of screws.

From Eqs.(15) and (16), mean value μ_p and standard deviation σ_p of scattering of axial tension become as follows.

$$\mu_p=\left(\frac{K_{\max}+K_{\min}}{2K_{\max}\cdot K_{\min}}+a\frac{K_{\max}-K_{\min}}{K_{\max}\cdot K_{\min}}\sin\theta_z\cdot\cos\theta_z\right)\times\frac{T_{\text{mean}}}{d} \quad (33)$$

$$\sigma_p=\left(\frac{K_{\max}-K_{\min}}{rK_{\max}\cdot K_{\min}}\sin\theta_z+\frac{a(K_{\max}+K_{\min})}{2rK_{\max}\cdot K_{\min}}\cos\theta_z\right)\times\frac{T_{\text{mean}}}{d} \quad (34)$$

As described above, it was also shown that the problem of the $Z=X/Y$ type can be analyzed by using a confidence limit ellipse, expressing it as $Z=Xy(y=1/Y)$, under the condition that the approximation can be made with such a linear relation obtained as the following equation.

$$y=c_1Y+c_2 \quad (35)$$

where c_1 and c_2 are constants.

The distribution of axial tension can be quantitatively shown by the characteristic values of tightening bolts which are determined experimentally.

An axial tension meter (Skidmore J-type Standard) was used for the measurement of the torque coefficient. The experimental conditions were selected by the two-way layout method using the lubricating condition and screw size as factors, as shown in Table 5. Table 6 shows the mean value K_{mean} and the 95 % confidence limit from the experimental results.

The actual confidence limit ellipse using the experimental results in Table 6 and taking the tool coefficient as $a=0.2$ is shown in Fig. 7.

4. Discussion

4.1 Effect of higher initial axial tension

It has been considered that the state of proper tightening of bolts gives sufficiently high axial tension so that no breaking or loosening takes place during operation. The advantage of applying this method to the control of the axial tension in bolts is that it allows us to give higher initial tension to bolts. As shown earlier in Fig. 6, the maximum value of axial tension by the conventional concept is indicated by point q ; the intersection of the maximum value of tightening torque and the minimum value of torque coefficient, and the axial tension has been thought to be apparently normally distributed. However, the point q is outside of the confidence ellipse obtained by this method and the result is that the actual confidence limit is higher than that which has been assumed. A comparison of this method and the conventional one is shown in Fig. 8. The confidence limit ellipse on a bolt-tightening diagram is schematically shown in Fig. 8(a), and details are shown in Fig. 8(b).

In the conventional method, the apparent distribution of the axial tension was between point q and point r, but in the case of using a confidence limit ellipse as the proposed method, it is actually a normal distribution between point s and point t. Point s is lower than point q, and has a margin for the yield load. Therefore, the proposed method can raise point s to point u, which reaches the same level as point q, and at this time, the lower limit point t is raised to point v. Thus, this method can obtain higher axial tension, as shown at the beginning.

4.2 Quantitative evaluation of conventional method and proposed method

Regarding the results of the application to the limit of transmitted torque for a disk clutch in section 3.1, a quantitative examination was carried out. In section 3.1, the 90 % confidence limit value was determined by using a confidence limit ellipse. On the other hand, in the conventional method, the limit of transmitted torque is determined by Eq.(36).

$$M_{c\max}=c'\mu_{c\max}F_{\max} \tag{36}$$

However, as shown in Fig.9, this point of $M_{c\max}$ exists on the confidence limit ellipse that indicates the boundary of a higher confidence limit value. This higher confidence limit ellipse through this point yields the Eq.(37).

$$\frac{(x-\mu_x)^2}{2A^2}+\frac{(y-\mu_y)^2}{2B^2}=1 \tag{37}$$

From the comparison of Eq.(10), the confidence limit value corresponds to $\sqrt{2}$ times higher at the percentile

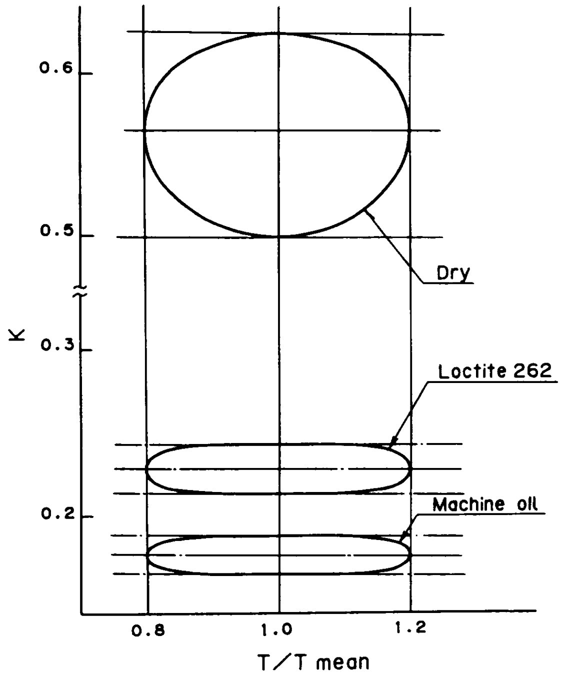

Fig. 7 Characteristic distribution by confidence limit

Table 5 Two-way layout test condition

Lubrication \ bolt	M12	M16
Dry	1−1	2−1
Machine oil	1−2	2−2
Loctite 262	1−3	2−3

Table 6 Experimental analysis for torque coefficient (mean valve and confidence limit)

Condition	Min	Mean	Max
Dry	0.498	0.564	0.630
Machine oil	0.164	0.176	0.188
Loctite 262	0.214	0.229	0.244

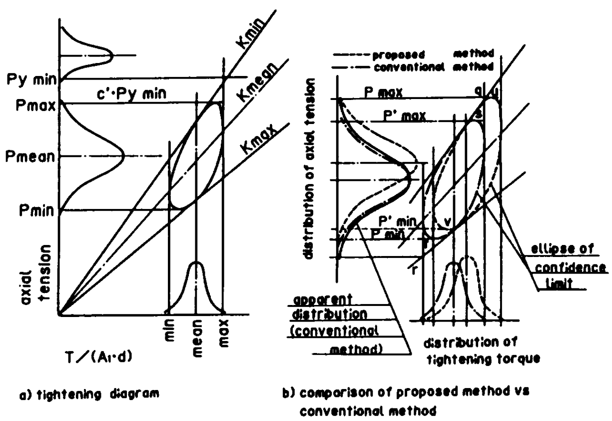

Fig. 8 Axial tension distribution

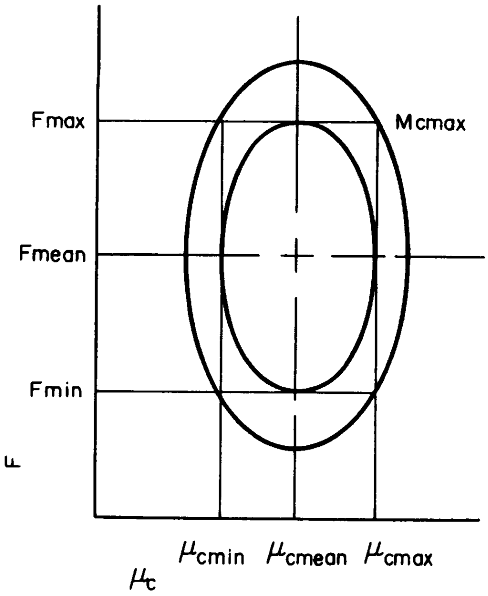

Fig. 9 Compare the comfidence limit

value in normal distribution. The relation is quantitatively determined and compared, and the results are shown in Table 7. As shown in the table, it is known that the conventional method resulted in an excessive evaluation.

5. Conclusions

The results can be summarized as follows.

1) The problem of the product of two independent probability variables which are normally distributed was theoretically analyzed using the confidence limit ellipse.

2) For this distribution of product z, the method for the calculation of probability P_f, when z did not exceed a limit, was shown.

3) As an example for the $z=x/y$ type problem, the limit of transmitted torque analysis in a disk clutch at the 90 % confidence limit was carried out, and a rational result was obtained by the proposed method.

Table 7 Comparison of limit of transmitted torque

	Mc max(N·m)	r	C.L.
Proposed method	248.7	1.645	90.0 %
Conventional method	279.0	2.326	98.0 %

C.L : confidence limit

4) For the $z=x/y$ type problem, the proposed method is applicable to the axial tension control on a bolted joint in the case in which the hyperbolic relation can be treated as linear.

5) From the result of the analysis by the proposed method, it was found that the maximum and minimum values in the scatter of the product z by the conventional method resulted in the unintentional use of a higher confidence limit level. The confidence limit level corresponded to $\sqrt{2}$ times greater in the percentile value in the standard normal distribution table.

References

(1) Yoshimoto, I., Trans. Jpn. Soc. Mech. Eng.,(in Japanese), Vol. 38, No. 315, Part 3(1973), p. 3017.

(2) Hareyama, S., A Study on Bolt Axial Tension Control by the Calibrated Wrench Method(1st Report, Distribution of Bolt Axial Tension), Trans. Jpn. Soc. Mech. Eng.,(in Japanese), Vol. 53, No. 495, C(1987), p. 2373.

(3) Hareyama, S. and Nakamura, T., A Study on Bolt Axial Tension Control by the Calibrated Wrench Method (2nd Report, Increase of Initial Axial Tension by the Ellipse of Confidence Limit), Trans. Jpn. Soc. Mech. Eng.,(in Japanese), Vol. 54, No. 508, C(1988), p. 3048.